Dynamic Conditional Correlation with Elliptical Distributions*

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Abstract

The Dynamic Conditional Correlation (DCC) model of Engle has made the estimation of multivariate GARCH models feasible for reasonably big vectors of securities' returns. In the present paper we show how Engle's multi-step estimation of the model can be easily extended to elliptical conditional distributions and apply different leptokurtic DCC models to twenty shares listed at the Milan Stock Exchange.

Keywords: Multivariate GARCH, Correlation, Elliptical distributions, Fat tails.
JEL codes: C32, C51, C87.

1 Introduction

As Robert Engle has remarked in his Nobel prize lecture, multivariate GARCH (MV-GRACH) models have been only partially successful despite their great potential usefulness. The reasons for this are two: i) the explosive growth of the number of parameters to be estimated compared to the number, k, of time series in the model, ranging from $O(k^4)$ in the unrestricted vech form (Bollerslev et al. 1988), to $O(k^2)$ in the standard BEKK (Engle and Kroner 1995) and in the diagonal vech, just to name the mostly cited MV-GARCH, ii) the difficulties of ensuring the positive definiteness of the conditional covariance matrices in many MV-GARCH models and the lack of interpretation of the constrains suited to this end.

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Bollerslev (1990) with his Constant Conditional Correlation GARCH (CCC) model and, more recently, Alexander (2001) with her Orthogonal GARCH (O-GARCH), have shown that a feasible way for estimating MV-GARCH models, applied to portfolios of realistic dimensions, is splitting the estimation in two steps, one of which is a sequential application of univariate GARCH models. Both of this models have, nevertheless, some drawbacks. The CCC model does not allow the correlations between securities to vary over time, and this restriction is not plausible in many situations. The O-GARCH model, consisting in the application of univariate GARCH models to time series, "orthogonalized" through Principal Component Analysis applied to unconditional sample correlations, may be effective but it is a "black box" technique, lacking of interpretation both for the coefficients and for the dynamics driving the conditional correlations.

The Dynamic Conditional Correlation MV-GARCH (DCC) model of Engle (2002) preserves the ease of estimation of the CCC model through a multi-step procedure, but allows for correlations to change over time. Furthermore, Engle and Sheppard (2001) derive the asymptotic distribution of the two stage estimates, making testing possible.

In the present work we show how the use of multivariate, fat tailed elliptical distributions may improve the fit of DCC models to the vector of returns of many real financial assets, when compared to the conditional Gaussian model. The elliptical DCC is then applied to twenty highly capitalized companies listed at the Milan Stock Exchange.

2 Review of elliptical distributions: definition and main properties

The m-dimensional random vector $\boldsymbol{X}$ is said to be distributed elliptically[1], symbolically $\boldsymbol{X} \sim \mathrm{EC}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$, if its characteristic function may be expressed in the form

$$\mathrm{E}[\exp(i\boldsymbol{t}'\boldsymbol{X})] = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\phi(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}),$$

with $\boldsymbol{\mu}$ m-dimensional vector, $\boldsymbol{\Sigma}$ positive definite $m \times m$ matrix, and $\phi(.)$ scalar function, referred to as *characteristic generator*.

We state without proof the principal properties of elliptical distributions, for a thorough treatment refer to Fang et al. (1990):

P1. if $X \sim \mathrm{EC}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ has a density, this has the form

$$f(\boldsymbol{x}) = c|\boldsymbol{\Sigma}|^{-\frac{1}{2}} g\Big((\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\Big)$$

with $g(.)$ a scalar function, referred to as *density generator* and the notation $X \sim \mathrm{EC}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$ may also be used;

[1] An alternative name for elliptical distributions is *elliptically contoured distributions.*

P2. suppose that $X \sim \mathrm{EC}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ possess k moments, if $k \geq 1$, then $\mathrm{E}(\boldsymbol{X}) = \boldsymbol{\mu}$, and if $k \geq 2$, then $\mathrm{Cov}(\boldsymbol{X}) = \gamma\boldsymbol{\Sigma}$, with $\gamma = -2\psi'(0)$;

P3. if $X \sim \mathrm{EC}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$, for any given $p \times m$ matrix $\boldsymbol{A}$ with rank $p \leq m$ and any p-dimensional vector $\boldsymbol{b}$

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{b} \sim \mathrm{EC}_p(\boldsymbol{A}\boldsymbol{\mu} + \boldsymbol{b}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}', \phi);$$

P4. if

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix} \sim \mathrm{EC}\left(\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}\right),$$

then

$$\boldsymbol{X}_1 | \boldsymbol{X}_2 \sim \mathrm{EC}\Big(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{X}_2 - \mu_2),\ \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21},\ \phi_{q(\boldsymbol{X}_2)}\Big),$$

where $\phi_{q(\boldsymbol{X}_2)}$ depends on the value assumed by $\boldsymbol{X}_2$ through the scalar-valued function $q(\boldsymbol{X}_2) = (\boldsymbol{X}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{X}_2 - \boldsymbol{\mu}_2)$;

P5. if we partition the vector $\boldsymbol{X}$ as above, then

$$\boldsymbol{X}_1 \sim \mathrm{EC}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, \phi).$$

Remarks:

1. notice that it is always possible to rewrite an elliptical distribution with second moments so that $\psi'(0) = -1/2$ and $\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma}$;
2. the linear correlation matrix

$$\boldsymbol{R} = \boldsymbol{D}^{-1}\boldsymbol{\Sigma}\boldsymbol{D}^{-1},$$

with $\boldsymbol{D}$ diagonal matrix with elements that are the square root of the elements on the diagonal of $\boldsymbol{\Sigma}$, can be sensibly defined even when the second moment does not exist;

3. it can be easily verified that the normal distribution, Student's t and Laplace distributions are members of the class of elliptical distribution.

3 The elliptical DCC model

Let $\boldsymbol{r}_t$ be a k-dimensional vector process adapted to the filtration $\mathcal{F}_t$ with conditional distribution given by

$$\boldsymbol{r}_t | \mathcal{F}_{t-1} \sim \mathrm{EC}_k(\boldsymbol{0}, \boldsymbol{\Sigma}_t, g), \tag{1}$$

where $\boldsymbol{\Sigma}_t$ is a positive definite $\mathcal{F}_{t-1}$-measurable dispersion matrix process defined by

$$\boldsymbol{\Sigma}_t = \boldsymbol{D}_t\boldsymbol{R}_t\boldsymbol{D}_t, \tag{2}$$

with $\boldsymbol{D}_t$ diagonal matrix given by the recursion[2]

$$\boldsymbol{D}_t^2 = \mathrm{diag}\{\boldsymbol{\omega}\} + \mathrm{diag}\{\boldsymbol{\kappa}\} \circ \boldsymbol{r}_{t-1}\boldsymbol{r}'_{t-1} + \mathrm{diag}\{\boldsymbol{\lambda}\} \circ \boldsymbol{D}_{t-1}^2, \tag{3}$$

$\circ$ representing element by element multiplication, and with $\boldsymbol{R}_t$, conditional correlation matrix defined by the set of equations

$$\begin{aligned} \boldsymbol{\xi}_t &= \boldsymbol{D}_t^{-1}\boldsymbol{r}_t \\ \boldsymbol{Q}_t &= \boldsymbol{S} \circ (\mathbf{1}\mathbf{1}' - \boldsymbol{A} - \boldsymbol{B}) + \boldsymbol{A} \circ \boldsymbol{\xi}_{t-1}\boldsymbol{\xi}'_{t-1} + \boldsymbol{B} \circ \boldsymbol{Q}_{t-1} \\ \boldsymbol{R}_t &= \mathrm{diag}\{\boldsymbol{Q}_t\}^{-\frac{1}{2}}\,\boldsymbol{Q}_t\,\mathrm{diag}\{\boldsymbol{Q}_t\}^{-\frac{1}{2}}. \end{aligned} \tag{4}$$

Equation (3) is just a set of univariate GARCH models with parameters ω_i, κ_i and λ_i, $(i = 1, \ldots, k)$, applied to every element of the vector $\boldsymbol{r}_t$. Equation (4) controls the dynamics of the conditional correlation matrix $\boldsymbol{R}_t$ through the square symmetric matrices of parameters $\boldsymbol{S}$, $\boldsymbol{A}$ and $\boldsymbol{B}$. Ding and Engle (2001) show that if $\boldsymbol{A}$, $\boldsymbol{B}$ and $(\mathbf{1}\mathbf{1}' - \boldsymbol{A} - \boldsymbol{B})$ are positive semi-definite and $\boldsymbol{S}$ is positive definite, then $\boldsymbol{Q}_t$ is also positive definite. In order to keep small the number of parameters to be simultaneously estimated, $\boldsymbol{A}$ and $\boldsymbol{B}$ are usually taken as scalars or set equal to $\boldsymbol{A} = \boldsymbol{\alpha}\boldsymbol{\alpha}'$ and $\boldsymbol{B} = \boldsymbol{\beta}\boldsymbol{\beta}'$, with $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ k-dimensional vectors of parameters. For the same reason, $\boldsymbol{S}$, which can be shown to be the unconditional correlation matrix, is estimated using the sample correlation of the standardized residuals $\boldsymbol{\xi}_t$.

If in equation (1) we take an elliptical distribution with density, then it is easy to build the log-likelihood function

$$l(\boldsymbol{\theta}) = \sum_{t=1}^{T} \left\{ \log c_m - \frac{1}{2}\log|\boldsymbol{\Sigma}_t| + \log g(\boldsymbol{r}_t\boldsymbol{\Sigma}_t^{-1}\boldsymbol{r}'_t) \right\}, \tag{5}$$

which, for a moderate number k of assets, may be maximized numerically. When the number of assets, and with it, the number of parameters is too large, then a three steps estimation procedure may be exploited to obtain consistent, asymptotically normal, although inefficient, estimates of the parameters.

1st step
Since the marginals of an elliptical distribution are elliptical distributions of the same family (property P2.), the parameters ω_i, κ_i and λ_i of the sequence of univariate GARCH models in equation (3) may be estimated by maximizing the k univariate likelihoods $\mathrm{EC}(0, \sigma_{ii}, g)$, for $i = 1, \ldots, k$. Through the recursion (3) the matrices $\boldsymbol{D}_t$ and the standardized residuals, $\boldsymbol{\xi}_t = \boldsymbol{D}_t^{-1}\boldsymbol{r}_t$ may be estimated.

2nd step
The sample covariance matrix of the standardized residuals estimated in the first step is then used as estimate of the matrix $\boldsymbol{S}$ in equation (4).

[2]The notation $\mathrm{diag}\{\boldsymbol{x}\}$ denotes a quadratic matrix with vector $\boldsymbol{x}$ on its principal diagonal and zeros elsewhere.

3rd step
Using the estimated $\boldsymbol{D}_t$ and $\boldsymbol{S}$, the likelihood

$$l(\boldsymbol{A}, \boldsymbol{B}) = \sum_{t=1}^{T} \left\{ \log c_m - \frac{1}{2} \log |\boldsymbol{R}_t| - \log |\hat{\boldsymbol{D}}_t| + \log g(\hat{\boldsymbol{\xi}}_t \boldsymbol{R}_t^{-1} \hat{\boldsymbol{\xi}}_t') \right\},$$

is maximized with respect to the parameters in $\boldsymbol{A}$ and $\boldsymbol{B}$ (usually the two scalars α and β).

Consistency and asymptotic normality of the 3-step estimates may be demonstrated exploiting the same results of Newey and McFadden (1994) used by Engle and Sheppard (2001). Let $\boldsymbol{\phi} = (\omega_1, \kappa_1, \lambda_1, \dots, \omega_k, \kappa_k, \lambda_k)'$ be the parameters' vector of the first step, $\boldsymbol{\rho} = (s_{1,2}, \dots, s_{1,k}, \dots, s_{k,1}, \dots, s_{k,k-1})'$ contain the unique elements of matrix $\boldsymbol{S}$, which are the 2nd step parameters, and $\boldsymbol{\psi} = (\alpha, \beta)'$ be the vector of the parameters estimated in the 3rd step. Furthermore, let[3]

$$\begin{aligned} \boldsymbol{h}^{(1)}(\boldsymbol{r}_t, \boldsymbol{\phi}) &= \nabla_{\boldsymbol{\phi}} \{ l_i(r_{i,t}, \omega_i, \kappa_i, \lambda_i) \}_{i=1,\dots,k} \\ \boldsymbol{h}^{(2)}(\boldsymbol{r}_t, \boldsymbol{\phi}, \boldsymbol{\rho}) &= \operatorname{vech}(\hat{\boldsymbol{\xi}}_t \hat{\boldsymbol{\xi}}_t{}' - \boldsymbol{S}) \\ \boldsymbol{h}^{(3)}(\boldsymbol{r}_t, \boldsymbol{\phi}, \boldsymbol{\rho}, \boldsymbol{\psi}) &= \nabla_{\boldsymbol{\psi}} l_c(\boldsymbol{r}_t, \boldsymbol{\phi}, \boldsymbol{\rho}, \boldsymbol{\psi}), \end{aligned}$$

where $l_i(r_{i,t}, \omega_i, \kappa_i, \lambda_i)$, for $i = 1, \dots, k$, is the t-th contribution to the log-likelihood of the i-th univariate GARCH model (1st step) and $l_c(\boldsymbol{r}_t, \boldsymbol{\phi}, \boldsymbol{\rho}, \boldsymbol{\psi})$ is the t-th contribution to the log-likelihood of the 3rd step. Letting $\boldsymbol{\theta} = (\boldsymbol{\phi}', \boldsymbol{\rho}', \boldsymbol{\psi}')'$, the 3-step procedure can be cast in GMM form with sample "orthogonality" conditions

$$\bar{\boldsymbol{h}}(\boldsymbol{\theta}) = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{h}(\boldsymbol{r}_t, \boldsymbol{\theta}) = \boldsymbol{0}$$

where

$$\boldsymbol{h}(\boldsymbol{r}_t, \boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{h}^{(1)}(\boldsymbol{r}_t, \boldsymbol{\phi}) \\ \boldsymbol{h}^{(2)}(\boldsymbol{r}_t, \boldsymbol{\phi}, \boldsymbol{\rho}) \\ \boldsymbol{h}^{(3)}(\boldsymbol{r}_t, \boldsymbol{\phi}, \boldsymbol{\rho}, \boldsymbol{\psi}) \end{bmatrix}$$

and the estimates are obtained by solving

$$\hat{\boldsymbol{\theta}}_T = \left\{ \boldsymbol{\theta} : \min_{\boldsymbol{\theta}} \bar{\boldsymbol{h}}(\boldsymbol{\theta})' \bar{\boldsymbol{h}}(\boldsymbol{\theta}) \right\}.$$

Since the system is just-identified with so many equations as parameters, the absolute minimum of the quadratic form (that is, 0) can be reached, and the orthogonality conditions relative to $\boldsymbol{h}^{(i)}$ are independent of those relative to $\boldsymbol{h}^{(i+j)}$ with j positive integer, this GMM estimate (or extremum estimate) is equivalent to the 3-step estimate.

[3] The vech operator is used with a slightly different definition than usual: it is here defined as the operator that stacks the elements below the diagonal of a square matrix.

Now let

$$\begin{aligned}
\boldsymbol{H}^{(1)}_{\boldsymbol{\phi}} &= \mathrm{E}\left[\nabla_{\boldsymbol{\phi}} \boldsymbol{h}^{(1)}(\boldsymbol{r}, \boldsymbol{\phi}_0)\right],\\
\boldsymbol{H}^{(2)}_{\boldsymbol{\phi}} &= \mathrm{E}\left[\nabla_{\boldsymbol{\phi}} \boldsymbol{h}^{(2)}(\boldsymbol{r}, \boldsymbol{\phi}_0, \boldsymbol{\rho}_0)\right],\\
\boldsymbol{H}^{(2)}_{\boldsymbol{\rho}} &= \mathrm{E}\left[\nabla_{\boldsymbol{\rho}} \boldsymbol{h}^{(2)}(\boldsymbol{r}, \boldsymbol{\phi}_0, \boldsymbol{\rho}_0)\right],\\
\boldsymbol{H}^{(3)}_{\boldsymbol{\phi}} &= \mathrm{E}\left[\nabla_{\boldsymbol{\phi}} \boldsymbol{h}^{(3)}(\boldsymbol{r}, \boldsymbol{\phi}_0, \boldsymbol{\rho}_0, \boldsymbol{\psi}_0)\right],\\
\boldsymbol{H}^{(3)}_{\boldsymbol{\rho}} &= \mathrm{E}\left[\nabla_{\boldsymbol{\rho}} \boldsymbol{h}^{(3)}(\boldsymbol{r}, \boldsymbol{\phi}_0, \boldsymbol{\rho}_0, \boldsymbol{\psi}_0)\right],\\
\boldsymbol{H}^{(3)}_{\boldsymbol{\psi}} &= \mathrm{E}\left[\nabla_{\boldsymbol{\psi}} \boldsymbol{h}^{(3)}(\boldsymbol{r}, \boldsymbol{\phi}_0, \boldsymbol{\rho}_0, \boldsymbol{\psi}_0)\right],
\end{aligned}$$

the expected Jacobian matrix is given by

$$\boldsymbol{H} = \mathrm{E}\left(\frac{\partial \boldsymbol{h}(\boldsymbol{r}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) = \begin{pmatrix} \boldsymbol{H}^{(1)}_{\boldsymbol{\phi}} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{H}^{(2)}_{\boldsymbol{\phi}} & \boldsymbol{H}^{(2)}_{\boldsymbol{\rho}} & \mathbf{0} \\ \boldsymbol{H}^{(3)}_{\boldsymbol{\phi}} & \boldsymbol{H}^{(3)}_{\boldsymbol{\rho}} & \boldsymbol{H}^{(3)}_{\boldsymbol{\psi}} \end{pmatrix} \tag{6}$$

By adapting from Newey and McFadden (1994, theorem 6.1), under regularity conditions[4]

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{D} \mathrm{N}(\mathbf{0}, \boldsymbol{H}^{-1}\boldsymbol{\Omega}\boldsymbol{H}^{-1}), \tag{7}$$

where

$$\boldsymbol{\Omega} = \mathrm{E}[\boldsymbol{h}(\boldsymbol{r}, \boldsymbol{\theta}_0)\boldsymbol{h}(\boldsymbol{r}, \boldsymbol{\theta}_0)']. \tag{8}$$

When opportune Laws of Large Numbers apply, consistent estimates of $\boldsymbol{H}$ and $\boldsymbol{\Omega}$ may be obtained by substituting expectations with sample means:

$$\hat{\boldsymbol{\Omega}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{h}(\boldsymbol{r}_t, \boldsymbol{\theta}_0)\boldsymbol{h}(\boldsymbol{r}_t, \boldsymbol{\theta}_0)',$$

and

$$\begin{aligned}
\hat{\boldsymbol{H}}^{(1)}_{\boldsymbol{\phi}} &= \frac{1}{T}\sum_{t=1}^{T}\left[\nabla_{\boldsymbol{\phi}} \boldsymbol{h}^{(1)}(\boldsymbol{r}_t, \boldsymbol{\phi}_0)\right],\\
&\dots\\
\hat{\boldsymbol{H}}^{(3)}_{\boldsymbol{\psi}} &= \frac{1}{T}\sum_{t=1}^{T}\left[\nabla_{\boldsymbol{\psi}} \boldsymbol{h}^{(3)}(\boldsymbol{r}_t, \boldsymbol{\phi}_0, \boldsymbol{\rho}_0, \boldsymbol{\psi}_0)\right],
\end{aligned}$$

as blocks of $\hat{\boldsymbol{H}}$.

[4] These condition are the standard conditions for the CAN property of general extremum estimators (Davidson 2000, White 1994, Gallant and White 1988, for example) and are rather difficult to verify in DCC-GARCH models.

4 The MultiGARCH object-class for Ox

Since the main advantage of the DCC-MVGARCH model over its competitors is the ease of estimation, even for a large number of assets, it is quite surprising that there is little applied work, in which the model is applied to portfolios of realistic size, in order to solve common financial problems such as optimal allocation and evaluation of the Value at Risk. This is probably due to the lack of a main-stream packages or software-libraries implementing the model[5].

In order to fulfill the promises of the DCC-MVGARCH and the practitioners' needs, we have written an object-class for Ox, which estimates DCC models with the 3-step procedure described above. At the moment the possible choices of conditional elliptical distributions are multivariate normal, multivariate Student's t[6]

$$f(\boldsymbol{r}_t|\mathcal{F}_{t-1}) = \frac{\Gamma\left[(\nu+m)/2\right]}{[\pi(\nu-2)]^{m/2}\,\Gamma(\nu/2)\,|\boldsymbol{\Sigma}_t|^{1/2}}\left[1+\frac{\boldsymbol{r}_t'\boldsymbol{\Sigma}_t^{-1}\boldsymbol{r}_t}{\nu-2}\right]^{-\frac{\nu+m}{2}},$$

and multivariate Laplace,

$$f(\boldsymbol{r}_t|\mathcal{F}_{t-1}) = \frac{2}{(2\pi)^{m/2}|\boldsymbol{\Sigma}_t|^{1/2}}\left(\frac{\boldsymbol{r}_t'\boldsymbol{\Sigma}_t^{-1}\boldsymbol{r}_t}{2}\right)^{\nu/2} K_\nu\left(\sqrt{2\boldsymbol{r}_t'\boldsymbol{\Sigma}_t^{-1}\boldsymbol{r}_t}\right).$$

where $K_\nu(.)$ is the modified Bessel function of third kind with index ν (Kotz et al. 2000, for instance).

We have tested the software using the daily log-returns of 20 high-capitalization shares listed in the Milan Stock Exchange, dating from January, the 1st 1999 to April, the 30th 2004. We have estimated the DCC-MVGARCH model with the three distributions trying different values for the Student's t degrees of freedom (DF). The normal and the Student's t DCC-MVGARCH (with at least 8.7 DF) converged quite quickly with relatively arbitrary starting points. When we used the conditional Laplace, even most of the univariate steps couldn't converge. Since the same problems were found using conditional Student's t with few DF, we have concluded that too leptokurtic densities and, thus, a too small number of tail observations, may make the likelihood too flat in a neighborhood of the maximum.

Table 1 reports the estimates and the log-likelihoods of the DCC-MVGARCH models with different conditional distributions (Student's t with a range of DF and Normal). According to the values of the log-likelihoods, the Student's t DCC-MVGARCH with 8.7 DF enjoys the best fit. On the other hand, the point estimates obtained in the different cases are not too different. However, giving a more accurate estimates of the tail-thickness (kurtosis) may be very important when assessing the riskiness of a portfolio of assets.

[5]When the first version of this paper was written the DCC model was implemented in RATS 5 exploiting only FIML estimation. The estimation could not converge for more then 3 or 4 simultaneous time series. In reviewing this article, we have apprehended from the Internet site of RATS, that version 6.2 contains the Gaussian multi-step estimation procedure as well. Since we have no access to this software we could not test it and compare it with ours.

[6]We use a version of the multivariate Student's t with covariance matrix $\boldsymbol{\Sigma}$, instead of $\frac{\nu}{\nu-2}\boldsymbol{\Sigma}$.

parameter	t(8.7)	t(8.8)	t(9)	t(10)	Normal
ω(ALLEANZA)	0.0779	0.0778	0.0778	0.0775	0.0651
κ(ALLEANZA)	0.1205	0.1203	0.1200	0.1189	0.1158
λ (ALLEANZA)	0.8651	0.8652	0.8652	0.8654	0.8798
ω(AUTOGRILL)	0.2375	0.2374	0.2374	0.2364	0.2826
κ(AUTOGRILL)	0.1602	0.1600	0.1598	0.1583	0.1628
λ (AUTOGRILL)	0.7983	0.7982	0.7981	0.7980	0.7842
ω(AUTOSTRADE)	0.1113	0.1110	0.1112	0.1111	0.1433
κ(AUTOSTRADE)	0.1716	0.1710	0.1705	0.1675	0.1456
λ (AUTOSTRADE)	0.7793	0.7798	0.7800	0.7821	0.8073
ω(FIDEURAM)	0.0903	0.0907	0.0916	0.0957	0.1695
κ(FIDEURAM)	0.0607	0.0607	0.0607	0.0606	0.0648
λ (FIDEURAM)	0.9277	0.9276	0.9274	0.9265	0.9147
ω(BNL)	0.2735	0.2735	0.2735	0.2717	0.3128
κ(BNL)	0.1129	0.1127	0.1123	0.1105	0.1063
λ (BNL)	0.8429	0.8430	0.8431	0.8440	0.8444
ω(BENETTON)	0.0530	0.0530	0.0530	0.0521	0.0558
κ(BENETTON)	0.0410	0.0410	0.0409	0.0403	0.0379
λ (BENETTON)	0.9441	0.9441	0.9442	0.9448	0.9506
ω(ENI)	0.0334	0.0335	0.0337	0.0347	0.0513
κ(ENI)	0.0577	0.0577	0.0575	0.0568	0.0527
λ (ENI)	0.9318	0.9318	0.9317	0.9315	0.9307
ω(FINMECCANICA)	0.0784	0.0786	0.0788	0.0804	0.1256
κ(FINMECCANICA)	0.0913	0.0912	0.0909	0.0899	0.0847
λ (FINMECCANICA)	0.9002	0.9002	0.9001	0.8999	0.8945
ω(GENERALI)	0.0660	0.0661	0.0663	0.0675	0.0960
κ(GENERALI)	0.1131	0.1130	0.1128	0.1120	0.1159
λ (GENERALI)	0.8700	0.8699	0.8698	0.8690	0.8547
ω(BANCA INTESA)	0.0902	0.0900	0.0897	0.0884	0.0848
κ(BANCA INTESA)	0.0917	0.0916	0.0914	0.0907	0.0900
λ (BANCA INTESA)	0.8972	0.8972	0.8973	0.8973	0.8980
ω(MEDIASET)	0.0603	0.0602	0.0601	0.0594	0.0595
κ(MEDIASET)	0.0636	0.0634	0.0631	0.0620	0.0570
λ (MEDIASET)	0.9291	0.9291	0.9292	0.9298	0.9336
ω(MEDIOBANCA)	0.0770	0.0769	0.0769	0.0767	0.0836
κ(MEDIOBANCA)	0.1436	0.1434	0.1431	0.1420	0.1521
λ (MEDIOBANCA)	0.8403	0.8404	0.8404	0.8407	0.8370
ω(MEDIOLANUM)	0.1446	0.1446	0.1445	0.1439	0.1441
κ(MEDIOLANUM)	0.0850	0.0850	0.0848	0.0843	0.0859
λ (MEDIOLANUM)	0.9001	0.9001	0.9000	0.8998	0.8986
ω(PIRELLI)	0.1050	0.1054	0.1060	0.1094	0.1644
κ(PIRELLI)	0.1373	0.1371	0.1366	0.1349	0.1309

λ (PIRELLI)	0.8417	0.8417	0.8419	0.8422	0.8501
ω(RAS)	0.0175	0.0175	0.0175	0.0177	0.0248
κ(RAS)	0.0653	0.0653	0.0652	0.0650	0.0726
λ (RAS)	0.9306	0.9306	0.9306	0.9303	0.9245
ω(SAIPEM)	0.3573	0.3574	0.3575	0.3581	0.4872
κ(SAIPEM)	0.1577	0.1573	0.1569	0.1545	0.1525
λ (SAIPEM)	0.7855	0.7857	0.7858	0.7867	0.7775
ω(SANPAOLO)	0.1360	0.1360	0.1360	0.1361	0.1646
κ(SANPAOLO)	0.0806	0.0805	0.0802	0.0789	0.0718
λ (SANPAOLO)	0.8983	0.8983	0.8984	0.8987	0.8996
ω(STM)	0.0553	0.0554	0.0556	0.0567	0.0743
κ(STM)	0.0594	0.0593	0.0591	0.0584	0.0557
λ (STM)	0.9387	0.9387	0.9387	0.9386	0.9386
ω(TELECOM)	0.0167	0.0167	0.0168	0.0172	0.0141
κ(TELECOM)	0.0532	0.0531	0.0529	0.0521	0.0409
λ (TELECOM)	0.9438	0.9438	0.9439	0.9442	0.9580
ω(TIM)	0.0177	0.0177	0.0178	0.0183	0.0285
κ(TIM)	0.0831	0.0830	0.0828	0.0818	0.0842
λ (TIM)	0.9168	0.9168	0.9169	0.9170	0.9130
α(DCC)	0.0053	0.0053	0.0054	0.0054	0.0056
β (DCC)	0.9862	0.9862	0.9862	0.9863	0.9880
log-likelihood	-54345.6	-54346.1	-54347.3	-54354.7	-55184.4

Table 1: Estimates of the DCC-MVGARCH model with 20 stocks for different conditional distributions (Student's t with DF in parenthesis and Normal).

The software allows to plot the estimated conditional variances, correlations and covariances as well. Figure 1 reports all the variances, while in figures 2 and 3 the covariances and correlations of ALLEANZA with all the other stocks are sketched. It is interesting to notice that starting form September, the 11$^{\text{th}}$ 2001 (observation 703), the correlations between almost all the stocks increase suddenly and remain high until the beginning of 2003. This underlines the fact that after the terrorist attacks of September 2001 the investors started giving more weight to international risk factors rather than to firm-specific information.

By looking at figure 3 we notice that the conditional correlations are less reactive to new shocks than conditional variances. This fact is also evident from the value of the ARCH parameter of the correlation equation ($\alpha = 0.0053$) which is much smaller then corresponding parameter (κ) in any of the conditional variance equation.

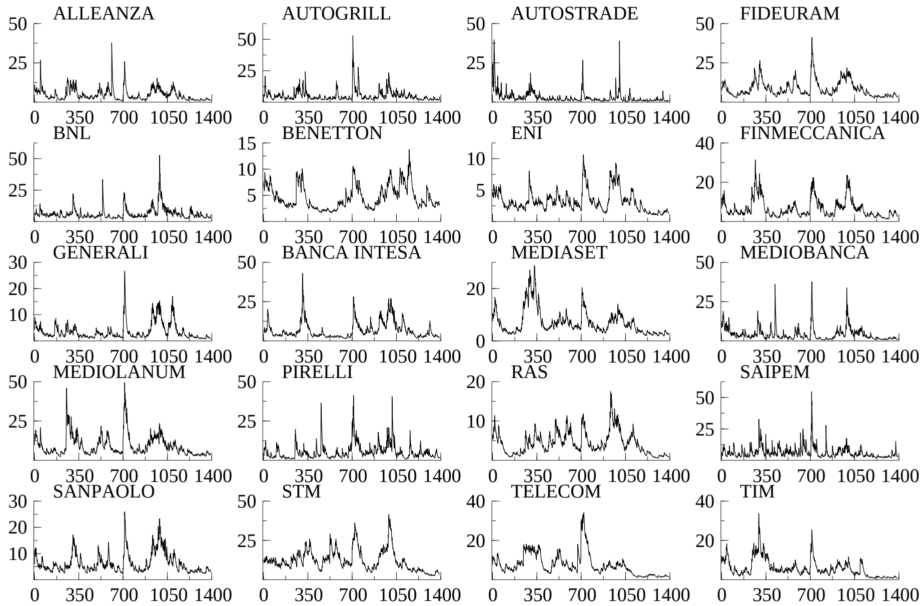

Figure 1: Estimated conditional variances for all the stocks.

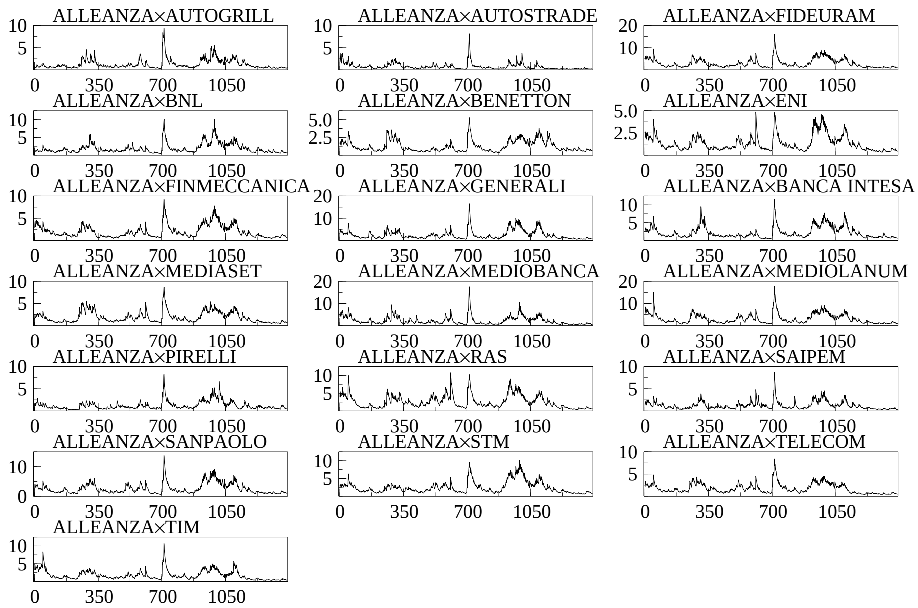

Figure 2: Estimated conditional covariances of ALLEANZA with all the other stocks.

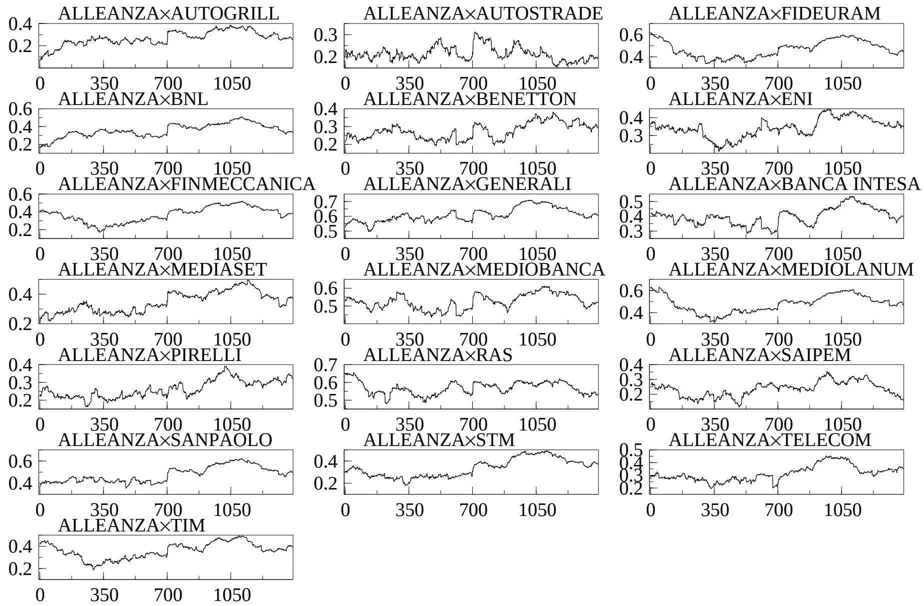

Figure 3: Estimated conditional correlations of ALLEANZA with all the other stocks.

5 Conclusion

We have proposed an extension of Engle's DCC model to the important family of elliptical distributions, which includes many thick-tailed densities and enjoy many of the properties of the Gaussian distribution. We have shown that the multi-step estimation procedure may be carried out for any choice of conditional elliptical distributions possessing a density, and stated the asymptotic properties of the estimator[7].

We have applied some elliptical DCC models to twenty shares of highly capitalized companies listed at the Milan Stock Exchange and shown that thick-tailed elliptical DCC models fit much better the Gaussian one.

An object-class for Ox for easily carrying out the estimation and plotting graphs similar to the ones shown in the paper has been written and left freely downloadable at the first author's web site[8].

[7]Without checking the validity of the underlying assumption for DCC-GARCH models, but research is this direction is still missing even fore the Gaussian DCC.

[8]http://www.statistica.unimib.it/utenti/p_matteo/

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